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Perturbation theory of Schrödinger operators in infinitely many coupling parameters

Manfred Requardt† and Anja Schlömerkemper‡

† Institut für Theoretische Physik, Universität Göttingen, Bunsenstrasse 9, 37073 Göttingen, Germany
‡ Max-Planck-Instiute for Mathematics in the Sciences, Inselstraße 22-26, 04103 Leipzig, Germany

E-mail: requardt@Theorie.Physik.Uni-Goettingen.de and Anja.Schloemerkemper@mis.mpg.de

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Abstract. In this paper we study the behaviour of Hamilton operators and their spectra which depend on infinitely many coupling parameters or, more generally, parameters taking values in some Banach space. One of the physical models which motivates this framework is a quantum particle moving in a more or less disordered medium. One may, however, also envisage other scenarios where operators are allowed to depend on interaction terms in a manner we are going to discuss below. The central idea is to vary the occurring infinitely many perturbing potentials independently. As a side aspect this then leads naturally to the analysis of a couple of interesting questions of a more or less purely mathematical flavour which belong to the field of infinite-dimensional holomorphy or holomorphy in Banach spaces. In this general setting we study in particular the stability of the self-adjointness of the operators under discussion and the analyticity of eigenvalues under the condition that the perturbing potentials belong to certain classes.

1. Introduction

The physical aim of this paper is the investigation of properties of Hamilton operators which depend on infinitely many coupling parameters, β_i, or perturbing potentials, V_i, i.e. we want to study Hamilton operators of the form

$$H(\beta) = H_0 + \sum_{i=1}^{\infty} \beta_i V_i \tag{1}$$

with some unperturbed Hamiltonian H_0, the properties of which are frequently assumed to be known. In typical cases H_0 is some free Hamiltonian such as, for example, the Laplacian $-\Delta$ or some relatively well behaved standard Hamiltonian of the form

$$H_0 = -\Delta + U \tag{2}$$

with a fixed interaction potential U.

In the general situation $\beta := (\beta_i)$ varies in a certain infinite-dimensional sequence space such as, for example, l^p with $1 \leqslant p \leqslant \infty$, where l^∞, i.e. the sequences which do not necessarily decay at infinity, is particularly interesting on physical grounds as we are primarily interested in perturbations which extend with the same strength to spatial infinity.

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The physical motivation to study this class of model Hamiltonians is the following: we envisage a quantum mechanical particle moving in an infinitely extended background consisting, say, of a more or less disordered array of atomic potentials. It is then an interesting scenario, both from a physical and a mathematical point of view, to test the response of the particle to independent variations of the coupling strengths or potentials making up the array. A typical case in point is a particle moving in a regular crystal which is then deformed or develops more or less irregularly distributed defects. Furthermore, possible applications to disordered media in general are obvious.

Among the various aspects one can or should investigate are several of a more mathematical or fundamental flavour such as, for example, *self-adjointness questions* or *analyticity properties of eigenvalues or the spectrum in general*. These are the problems we will mainly address in the following in order to set the stage as we are presently not aware whether such questions have been dealt with in this generality in the past. Let us note in this context that up to now the main thrust of investigations has rather gone into the study of *random Hamiltonians*, an approach which is in some sense complementary to the one we will develop in this paper.

In addressing problems of this general kind one soon realizes that intricate mathematical questions do emerge which do not belong to the standard arsenal of mathematical physics such as, for example, *infinite-dimensional holomorphy* or *holomorphy on Banach spaces*. Furthermore, one has to deal with Taylor series having a countable infinity of independent variables. In other words, this is yet another example how a natural physical problem quickly leads into some very advanced fields of pure mathematics.

This suggests the following organization of the paper. As a warm-up exercise we will treat in the next section the special case of bounded perturbations within the context of general operator theory and develop a couple of useful mathematical tools and concepts. This abstract approach, that is, making only very general assumptions on the class of potentials under discussion, leads, perhaps surprisingly, to certain technical problems if one attempts to apply it in a next step to unbounded operators. These problems are briefly discussed at the beginning of section 3.

In section 3 we then show that a seemingly appropriate concrete class of potentials is the so-called *Stummel class* if one is willing to adopt a more concrete setting, i.e. working within a concrete Hilbert space of functions and studying concrete Hamilton operators. We then return to more abstract considerations and give a brief review of *infinite-dimensional holomorphy* or *complex analysis in Banach spaces*. This allows one to treat Hamiltonians, which depend on infinitely many independent coupling constants (or more generally, coupling parameters belonging to some Banach space), and in particular their *perturbation theory* in a more systematic way in section 5.

2. Concepts and tools

The first main step consists in providing criteria so that the Hamiltonian $H = H_0 + \sum \beta_i V_i$ is again self-adjoint given the self-adjointness of H_0. To begin with, this problem shall be studied with the help of a simple class of perturbations for which the well known additional technical intricacies of the more general situation are expected to be absent.

Assumption 2.1. *Let H_0 be a self-adjoint operator on the Hilbert space $\mathcal{H}$. The V_i, $i \in \mathbb{N}$, are assumed to be uniformly bounded, i.e.*

$$\|V_i\| \leqslant v < \infty \tag{3}$$

for all i and some $v \in \mathbb{R}$.

The problem is to guarantee that the infinite sum over the potentials is again a well defined operator and, in this example, a bounded operator. In general it may easily happen that such an infinite sum is no longer defined on certain vectors in the Hilbert space (e.g. if the potentials tend to cluster too much around certain points in coordinate space). In order to prevent this one has to take some precautions. A sufficient condition which furthermore has a clear geometric or physical meaning is the following:

Definition 2.2 (Finite intersection property). *Let the V_i's be linear operators on $\mathcal{H}$. We say they have the finite intersection property if the following holds:*
There exists a projection-valued probability measure P on $\mathcal{B}(\mathbb{R}^m)$, the Borel σ-algebra over $\mathbb{R}^m$, so that:

(a) For each V_i exists a Borel set Ω_i with $P_{\Omega_i} V_i = V_i$.
(b) For each $i \in \mathbb{N}$ let I_i be the index set $\{i \neq j \in \mathbb{N} : \Omega_i \cap \Omega_j \neq \emptyset\}$. Then $\#(I_i)$ is uniformly bounded in i by some $n_0 < \infty$.

Remark. Note that in the general case the V_i can be almost arbitrary localized operators acting in some abstract Hilbert space. In the same sense the correlation of the projectors with certain sets in some space $\mathbb{R}^m$ can, while of course being physically motivated, be fairly indirect from a mathematical point of view.

The above entails that to each given V_i there exist at most n_0 projectors P_j (shorthand for P_{Ω_j}) so that $P_j V_i \neq 0$ since $P_j P_i = 0$ if $\Omega_j \cap \Omega_i = \emptyset$. By the same argument there exist at most n_0 potentials V_i so that $P_j V_i \neq 0$.

Example 2.3 (Multiplication operators). *If V_i are multiplication operators on $L^2(\mathbb{R}^m)$ they have the finite intersection property if*

(a) the support of V_i is contained in a Borel set Ω_i and
(b) if for these sets condition 2 of definition 2.2 holds with the P_i's being indicator functions.

Remark. The above condition entails that the potentials are sufficiently scattered in coordinate space. This can also be enforced by slightly different conditions such as the following. Assume that, given an arbitrary $x \in \mathbb{R}^m$ and a ball around x with some fixed diameter, only uniformly finitely many potentials meet this ball when x varies over $\mathbb{R}^m$. We will come back to this variant in section 3.

We now proceed as follows. With the help of the polar decomposition of V_i we have

$$\|V_i\psi\| = \|U_i|V_i|\psi\| = \||V_i|\psi\| \tag{4}$$

with $|V_i| := (V_i^* V_i)^{1/2}$. Applying the corresponding projector P_i to a self-adjoint V_i we obtain

$$P_i V_i = V_i = V_i^* = V_i P_i \tag{5}$$

hence

$$\||V_i|\psi\|^2 \leqslant v^2 \cdot \|P_i\psi\|^2 \tag{6}$$

and

$$|V_i| \leqslant v \cdot P_i \tag{7}$$

since for positive operators $A^2 \leqslant B^2$ implies $A \leqslant B$ (meaning $(\psi|A\psi) \leqslant (\psi|B\psi)$). For finitely many V_i it then follows that

$$\sum_1^n |V_i| \leqslant v \sum_1^n P_i. \tag{8}$$

Given the sequence of potentials V_i or projectors P_i we now make a disjoint refinement $\{\Omega'_j\}$ of the class of sets $\{\Omega_i\}$. This then yields a corresponding refinement of the class of projectors which are now orthogonal by construction (projection-valued measure, hence $\Omega'_i \cap \Omega'_j = \emptyset$ implies $P'_i \cdot P'_j = 0$). The construction is accomplished in the following way. First of all we can restrict ourselves to an arbitrary but fixed candidate of the class $\{\Omega_i\}$ which we for convenience call Ω_0. We then make the following definition.

Observation/Definition 2.4. *For each given $x \in \Omega_0$ there exists a unique maximal index set $I_x \subset \{0, 1, \ldots, k\}$, where $\Omega_1, \ldots, \Omega_k$ are the sets intersecting the start set Ω_0 and $x \in \Omega_j$ for $j \in I_x$. We call the subset of elements of Ω_0 having the same maximal index set I, Ω_I. This construction yields a (disjoint) partition into a finite number of Borel sets of the start set Ω_0. In the same way we can proceed with the other sets of the class $\{\Omega_i\}$, thus arriving at the disjoint partition $\{\Omega'_j\}$.*

Remark. Note that the sets of the refinement basically consist of certain intersections and corresponding complements within the class of the initial sets.

Lemma 2.5. *The sets Ω'_j represent a disjoint partition of the original class with a given Ω_i being resolved into at most 2^{n_0} disjoint sets where n_0 was the upper bound on the number of sets Ω_j intersecting a given Ω_i. By the same token we obtain a resolution into mutually orthogonal projectors with*

$$P_i = \sum_1^{k_i} P'_j \qquad k_i \leqslant 2^{n_0}. \tag{9}$$

Proof. By assumption at most n_0 sets can intersect our arbitrarily chosen Ω_0. Furthermore, one of the disjoint sets can result from the maximal index set $\{0\}$, corresponding to the complement in Ω_0 of the union of all the intersections of Ω_i with Ω_0. The optimal scenario can then be estimated by counting the number of subsets of a (n_0)-set (i.e. having n_0 elements), which is 2^{n_0}. To this we have to add 1 for the above complement and subtract 1 for the empty set counted in 2^{n_0}. This proves the above estimate. □

For a finite sum of P_i's we then have

$$\sum_1^n P_i \leqslant n_0 \cdot \sum_j^k P'_j \tag{10}$$

with k being a number between n and $2^{n_0} \cdot n$. Note that each P'_j can occur at most n_0 times on the right-hand side. By construction $\sum_1^k P'_j$ is again a projector (in contrast to the left-hand side), and hence has norm one and we obtain (cf equation (8)):

$$\sum_1^n |V_i| \leqslant v \cdot \sum_1^n P_i \leqslant v \cdot n_0 \cdot \mathbf{1} \tag{11}$$

implying

$$\left\| \sum_1^n |V_i| \right\| \leqslant v \left\| \sum_1^n P_i \right\| \leqslant v \cdot n_0. \tag{12}$$

With $|V_i|$ positive the sum on the left-hand side is monotonically increasing with a global norm bound given by the right-hand side. As for self-adjoint V_i $|(\psi|V_i\psi)| \leqslant (\psi||V_i|\psi)$ holds, we obtain

$$\left|\left(\psi\left|\sum_1^n V_i\psi\right.\right)\right| \leqslant \left(\psi\left|\sum_1^n |V_i|\psi\right.\right) \tag{13}$$

and

$$\left(\psi|\sum_1^n |V_i|\psi\right) \leqslant \|\psi\| \cdot \left\|\sum_1^n |V_i|\psi\right\| \leqslant v \cdot n_0 \cdot \|\psi\|^2. \tag{14}$$

Thus the sequence $\sum_1^n V_i$ converges weakly to a bounded operator $V = \sum_1^\infty V_i$ because the left-hand side is a Cauchy sequence.

Conclusion 2.6. *(a) Under the assumptions made above $\sum_1^\infty V_i$ can be defined as a weak limit and is again a self-adjoint bounded operator*

(b) This implies by standard reasoning (see, for example, [ReSi1] or [Ka]) that $H = H_0 + \sum V_i$ is again a self-adjoint operator on the domain of H_0 where here and in the following, unless otherwise noted, unspecified summation always means summation from 1 *to* ∞.

Corollary 2.7. *This applies, in particular, to a potential*

$$V = \sum \beta_i V_i \qquad \beta_i \in \mathbb{R} \qquad \beta := (\beta_i) \in l^\infty(\mathbb{R}) \qquad V_i \text{ as above}. \tag{15}$$

Remark 2.8. *(a) If the V_i are real multiplication operators on some $L^2(\mathbb{R}^m)$ the above sum can be shown to converge even in the strong sense. The underlying abstract reason for this stronger property lies in the fact that now all the V_i do automatically commute which allows for certain technical manipulations of sums which do not seem to be possible in the more general case (the proof can be found in [Sc]).*

(b) Furthermore, in this concrete case there exists a more straightforward variant on the above proof. The operator bound of V_i is the ess sup $|v_i(x)|$. *By assumption at most n_0 v_i meet at a point x. The operator norm of $\sum V_i$ is hence bounded by $v \cdot n_0$.*

Warning. In the generic case the above convergence is not in operator norm. Assuming, for example, that $\|V_i\| \geqslant \varepsilon > 0$ for all i, the sequence of sums $\sum_1^n V_i$ is evidently not a Cauchy sequence in norm.

In the typical physical situation the occurring potentials are frequently not of (strictly) finite range but decay at infinity with a certain rate. The *finite intersection property* introduced above emulates to some extent such a *finite-range condition*. It is therefore an interesting question as to what extent an infinite range of the potentials under discussion can be admitted. We make the following assumption.

Assumption 2.9 (Infinite range). *We assume that each potential V_i can be decomposed as*

$$V_i = V_i^a + V_i^b \tag{16}$$

with V_i^a fulfilling the finite intersection property. We assume further that with the help of the methods developed in this paper

$$H' = H_0 + \sum V_i^a \tag{17}$$

can be given a rigorous meaning as a self-adjoint operator. We want to impose a condition on V_i^b so that

$$\left\| \sum V_i^b \right\| < \infty \tag{18}$$

i.e. so that

$$H = H_0 + \sum V_i^a + \sum V_i^b \tag{19}$$

is a well defined self-adjoint operator where $\sum V_i^b$ is a bounded perturbation of H'. On the other hand, V_i^b need not fulfil the finite intersection property. We assume that V_i are multiplication operators in $L^2(\mathbb{R}^m)$ with V_i^b centred around points R_i and decaying in the following way:

$$|V_i^b(x)| \leqslant constant/(1 + |R_i - x|)^k \tag{20}$$

for some $k > 0$ and with the R_i distributed in $\mathbb{R}^m$ according to

$$|R_i - R_j| > 2A > 0 \tag{21}$$

if $i \neq j$.

Our strategy is to show that under these conditions

$$\sum |V_i^b(x)| \leqslant \sum \text{constant}/(1 + |R_i - x|)^k \leqslant B < \infty \tag{22}$$

for all $k \geqslant k_0$. To this end we prove the following simple lemma.

Lemma 2.10. *With $x \in \mathbb{R}^m$ and $K_R(x)$ a ball of radius R centred at x there are at most $(R + A)^m/A^m$ points R_i in $K_R(x)$ if $|R_i - R_j| > 2A$. Correspondingly, one can estimate the number of points in a spherical shell $K_{[R,R+d)}$ around x of radii R, $R + d$ with $R > A$. We have*

$$\#(R_i) \leqslant [(R + d + A)^m - (R - A)^m]/A^m. \tag{23}$$

Proof. Let $\{R_i\}$ be the set of points lying in $K_R(x)$. Draw a sphere of radius A around each R_i. The corresponding balls do not intersect and we can hence estimate

$$\#(R_i) \cdot c_m \cdot A^m \leqslant c_m \cdot (R + A)^m \tag{24}$$

where c_m is the volume of the unit sphere in $\mathbb{R}^m$. From this we can conclude

$$\#(R_i) \leqslant (R + A)^m/A^m. \tag{25}$$

In the same way we prove the second statement. □

We can now proceed as follows:

$$\sum |V_i^b(x)| = \sum_{R_i \in K_l} |V_i^b(x)| + \sum_{n \geqslant l} \sum_{R_i \in K_{[n,n+1)}} |V_i^b(x)| \tag{26}$$

where $n, l \in \mathbb{N}$ with l arbitrary but fixed so that $l > A$. The right-hand side can be estimated so that

$$\sum |V_i^b(x)| \leqslant C_0 + \sum_{n \geqslant l} (\text{constant}/(1+n)^k)\,[(n+1+A)^m - (n-A)^m]/A^m$$
$$\leqslant C_0' + C_1 \sum_n n^{(m-1-k)} \tag{27}$$

where C_0, C_0' and C_1 are constants independent of the point x (note that the leading n^m-powers vanish. Furthermore, we have absorbed sums over terms with a smaller power than $m-1-k$ in the constants). This sequence is convergent for $k > m$, hence:

Observation 2.11. *For potentials fulfilling the criteria of assumption 2.9 the sum over infinite range potentials,* $\sum V_i^b$, *yields a bounded operator if*

$$|V_i^b(x)| \leqslant \textit{constant}/(1+|R_i - x|)^k \qquad k > m \tag{28}$$

where m *is the space dimension.*

3. The Stummel class

It is tempting to try to proceed in the same abstract way as developed in section 2 by simply admitting more general classes of potentials or operators V_i. Our original idea was to employ the famous criterion of *Kato smallness* in its abstract form in order to arrive at self-adjoint perturbations of a given self-adjoint start Hamiltonian (see, for example, [Ka, ReSi2]).

Definition 3.1. *Let* H_0, V *be operators on* $\mathcal{H}$. V *is called* H_0*-bounded with relative bound* a *if*

(a) $D(V) \supset D(H_0)$
(b) $\|V\psi\| \leqslant a'\|H_0\psi\| + b\|\psi\|$

with a', b *non-negative and* a *understood as the infimum of such* a'.

Remark. A corresponding condition can be formulated in the *weak* (i.e. *form*) sense.

Theorem 3.2. *With* H_0, V *as above* $H = H_0 + V$ *is a closed or self-adjoint operator on* $D(H_0)$ *if* H_0 *is closed or self-adjoint and* V *is symmetric in the latter case provided that* $a < 1$.

It would now be natural to assume the V_i to be *Kato-small* in the above sense and then try to show the same for $\sum_{i=1}^{\infty} V_i$. However, to our surprise, irrespective of the direction of attack an approach along these abstract lines has not yet been successful due to technical intricacies in the manipulation and interchange of (infinite) sums and norm estimates. As a consequence we choose, for the time being, a more concrete approach in this section and consider a certain (in fact large) class of admissible potentials on some $L^2(\mathbb{R}^m)$.

Definition 3.3 (Stummel class). *With* v *a measurable function on* $\mathbb{R}^m$ *with respect to the standard Lebesgue measure we define for each* $\rho \in \mathbb{R}$

$$M_{v,\rho}(x) = \begin{cases} \left(\displaystyle\int_{|x-y|\leqslant 1} |v(y)|^2 |x-y|^{\rho-m}\, \mathrm{d}^m y \right)^{1/2} & \rho < m \\ \left(\displaystyle\int_{|x-y|\leqslant 1} |v(y)|^2\, \mathrm{d}^m y \right)^{1/2} & \rho \geqslant m. \end{cases} \tag{29}$$

The corresponding Stummel class is given by

$$M_\rho(\mathbb{R}^m) := \left\{ v : \mathbb{R}^m \to \mathbb{C} \;:\; \sup_{x \in \mathbb{R}^m} M_{v,\rho}(x) := M_{v,\rho} < \infty \right\}. \tag{30}$$

This class was introduced by Stummel in [St]. A textbook treatment can be found in, for example, [We]. Its properties have also been exploited in various papers by Simon (see, for example, [Si]).

Lemma 3.4 (without proof). *$M_\rho(\mathbb{R}^m)$ is a vector space and $M_{\rho_1} \subset M_{\rho_2}$ for $\rho_1 \leqslant \rho_2$.*

Example 3.5.

$$L^2(\mathbb{R}^m) + L^\infty(\mathbb{R}^m) \subset M_\rho(\mathbb{R}^m) \qquad L^2_{loc}(\mathbb{R}^m) \subset M_\rho(\mathbb{R}^m) \tag{31}$$

for $\rho \geqslant m$.

As to the reason for the choice of this particular class we would like to make some comments. Typically, mathematical physicists are accustomed to atomic potentials which consist of a *singular part*, having a few singularities of a certain degree away from infinity and perhaps a certain decaying tail extending to infinity. The left-hand side of example 3.5 is a typical case in point. These are the classes for which a lot of estimates can be found in the literature (see, for example, [ReSi2]) and which lead to a whole bunch of self-adjointness criteria. The potentials we want to discuss, however, are of a more intricate type. In our scenario the singularities extend generically to infinity as the particle is assumed to move in an infinitely extended (disordered) medium. As far as we can see, most of the standard estimates do apply only to the above-mentioned simpler class of atomic potentials (at least without modifications). On the other hand, as can be seen from definition 3.1, the Stummel condition is essentially a local estimate, that is, it is relatively insensitive to the number of singularities and their position in space. Therefore, it seems to be more suitable for our purposes at the moment.

We remarked already in section 2 that there exist variants on the *finite intersection property* given in definition 2.2 which may turn out to be more suitable in specific contexts. This is the case for the Stummel class.

Definition 3.6 (Variant on the finite intersection property). *Given $x \in \mathbb{R}^m$ and a ball around x with radius one, there are only uniformly finitely many potentials V_i (with respect to each x) which meet this ball. The bound being denoted by n_1.*

Remark. For *well behaved sets* Ω_i or supports of V_i all these conditions are essentially equivalent. On the other hand, there may be extreme situations where the one or the other turns out to be better adapted.

Theorem 3.7. *With $v_i(x)$ in $M_\rho(\mathbb{R}^m)$ for all i so that $M_{v_i,\rho} < \infty$ uniformly in i and $\{v_i\}$ fulfilling the intersection property in the sense of definition 3.6*

$$\sum_1^\infty \beta_i v_i \in M_\rho(\mathbb{R}^m) \tag{32}$$

holds for $\beta \in l^p(\mathbb{C})$, $1 \leqslant p \leqslant \infty$.

Proof. By assumption there exists a uniformly finite index set, J_x, for each x so that

$$\operatorname{supp}\left(\sum_{i=1}^\infty \beta_i v_i\right) \cap \{y \in \mathbb{R}^m : |x-y| \leqslant 1\}$$
$$= \operatorname{supp}\left(\sum_{i\in J_x} \beta_i v_i\right) \cap \{y \in \mathbb{R}^m : |x-y| \leqslant 1\}. \tag{33}$$

For $\rho < m$ we have

$$
\begin{aligned}
\int_{|x-y|\leqslant 1} \left| \sum_{i=1}^{\infty} \beta_i v_i(y) \right|^2 |x-y|^{\rho-m}\, \mathrm{d}^m y
&= \int_{|x-y|\leqslant 1} \left| \sum_{i\in J_x} \beta_i v_i(y) \right|^2 |x-y|^{\rho-m}\, \mathrm{d}^m y \\
&= \int_{|x-y|\leqslant 1} \left| \sum_{i\in J_x} \beta_i v_i(y) |x-y|^{(\rho-m)/2} \right|^2 \mathrm{d}^m y \\
&\leqslant \left(\sum_{i\in J_x} \left(\int_{|x-y|\leqslant 1} \left|\beta_i v_i(y) |x-y|^{(\rho-m)/2}\right|^2 \mathrm{d}^m y \right)^{1/2} \right)^2
\end{aligned}
\tag{34}
$$

where in the last inequality the Minkowski or triangle inequality for L^2 has been exploited. In a second step we obtain

$$
\begin{aligned}
&\left(\sum_{i\in J_x} \left(\int_{|x-y|\leqslant 1} \left|\beta_i v_i(y) |x-y|^{(\rho-m)/2}\right|^2 \mathrm{d}^m y \right)^{1/2} \right)^2 \\
&\qquad \leqslant (\sup_{i\in J_x} |\beta_i|)^2\, n_1^2 \left(\max_{i\in J_x} M_{v_i,\rho} \right)^2 \\
&\qquad \leqslant \|\beta\|_p^2\, n_1^2 \left(\max_{i\in J_x} M_{v_i,\rho} \right)^2 \\
&\qquad < \infty
\end{aligned}
\tag{35}
$$

uniformly in x as $\sup_{i\in J_x} |\beta_i| \leqslant \|\beta\|_\infty \leqslant \|\beta\|_p$. Analogously, one shows for $\rho \geqslant m$:

$$
\begin{aligned}
\int_{|x-y|\leqslant 1} \left| \sum_{i=1}^{\infty} \beta_i v_i(y) \right|^2 \mathrm{d}^m y = \int_{|x-y|\leqslant 1} \left| \sum_{i\in J_x} \beta_i v_i(y) \right|^2 \mathrm{d}^m y
&\leqslant \|\beta\|_p^2\, n_1^2 \left(\max_{i\in J_x} M_{v_i,\rho} \right)^2 \\
&< \infty
\end{aligned}
\tag{36}
$$

uniformly in x, which proves the statement. □

In the following we choose $H_0 = -\Delta$. The Laplace operator is self-adjoint on the Sobolev space $W_2(\mathbb{R}^m)$ (see, e.g., [ReSi2]). Furthermore, it can be inferred from slightly more general results provided in [We] that potentials from the Stummel class with $\rho < 4$ are defined on $W_2(\mathbb{R}^m)$ and are *relatively bounded* with respect to $-\Delta$ with *relative bound* zero.

Theorem 3.8. *Let $\|\cdot\|$, $\|\cdot\|_2$ be the (L^2) Hilbert space and Sobolev norm, respectively. For $\rho < 4$ there exists a constant $C \geqslant 0$ so that*

$$
\|v\psi\| \leqslant C M_{v,\rho} \|\psi\|_2 \qquad \forall\, v \in M_\rho(\mathbb{R}^m) \quad \psi \in W_2(\mathbb{R}^m). \tag{37}
$$

Furthermore, for all $\eta > 0$ there exists a C_η so that

$$
\|v\psi\| \leqslant \eta \|\psi\|_2 + C_\eta \|\psi\| \qquad \forall \psi \in W_2(\mathbb{R}^m). \tag{38}
$$

As the above Sobolev norm is equivalent to the graph norm of the Laplacian it follows that V is $-\Delta$-bounded with relative bound zero.

Consequences 3.9. *Under the above assumptions* $H(\beta) = -\Delta + \sum_{i=1}^{\infty} \beta_i V_i$ *is a closed, respectively, self-adjoint operator on* $W_2(\mathbb{R}^m)$.

A slight extension then yields

Theorem 3.10. *Under the above assumptions* $H(\beta) = -\Delta + U + \sum_{i=1}^{\infty} \beta_i V_i$ *is a closed or self-adjoint operator on* $W_2(\mathbb{R}^m)$ *if* U *is* $-\Delta$*-bounded with relative bound zero. In the latter case* U *and* V_i *have to be symmetric and* β_i *have to be real.*

So far the results on closedness or self-adjointness of Hamilton operators and the corresponding classes of admissible potentials. In the next two sections we are going to establish a theory of *analytic perturbation of spectra and operators* taking place in *infinitely many variables* at a time or variables varying in a general Banach space upon this groundwork.

4. Complex analysis in Banach spaces

In the second part of this paper we want to discuss analyticity properties of eigenvalues of Hamilton operators $H(\beta) = H_0 + \sum_{i=1}^{\infty} \beta_i V_i$ with H_0 some unperturbed Hamiltonian and $\beta_1, \beta_2, \ldots \in \mathbb{C}$. To do so one needs the notion of infinite-dimensional holomorphy.

Instead of dealing with infinitely many coupling parameters we will frequently regard $\beta = (\beta_1, \beta_2, \ldots)$ as an element of a Banach space, such as, for example, l^∞ and hence investigate perturbations in one Banach space-valued coupling parameter. While complex analysis in one complex variable belongs to the standard repertoire of the ordinary perturbation theory of operators we have to generalize it in the way described above to complex analysis in Banach spaces.

As this is perhaps not so widely known we summarize definitions and theorems which will be important in this enterprise. Many of the results can already be found in [HiPh]. As to more recent representations see, for example, [Mu] or [Ze].

In what follows X and Y are infinite-dimensional complex Banach spaces and $U \subset X$ is an open set. One of the difficulties of complex analysis in Banach spaces is a suitable definition of power series and differentiability.

Definition 4.1. *A formal power series from* X *to* Y *at* $a \in X$ *is a series of symmetric,* m*-linear mappings* $A_m : X^m \to Y$ *of the form*

$$\sum_{m=0}^{\infty} A_m (x-a)^m \tag{39}$$

with $A_m(x-a)^m := A_m(\underbrace{x-a, \ldots, x-a}_{m\text{-times}})$.

The radius of convergence of the power series $\sum_{m=0}^{\infty} A_m (x-a)^m$ is the supremum of all $r \geqslant 0$ so that the series converges uniformly in the closed ball $\bar{B}(a, r)$. In analogy to the formula of Cauchy–Hadamard the radius of convergence R is given by

$$\frac{1}{R} = \limsup_{m\to\infty} \|A_m\|^{1/m} \tag{40}$$

with $\|A_m\| := \sup_{\|x_1\| = \cdots = \|x_m\| = 1} \|A_m(x_1, \ldots, x_m)\|$ and $1/0 := \infty$ as well as $1/\infty := 0$. The series converges absolutely and uniformly in $\bar{B}(a, r)$ if $0 \leqslant r < R$.

With this notion of power series it is possible to introduce analytic mappings in Banach spaces.

Definition 4.2. *A map $f : U \to Y$ is called analytic if for each $a \in U$ there exists a ball $B(a, r) \subset U$ and a sequence of symmetric, m-linear, continuous mappings $A_m : X^m \to Y$ so that*

$$f(x) = \sum_{m=0}^{\infty} A_m (x - a)^m \tag{41}$$

for all $x \in B(a, r)$.

The sequence of mappings A_m is uniquely determined by f and a. We will frequently suppress the explicit dependence on f or a and set

$$A^m f(a) := A_m. \tag{42}$$

$$f(x) = \sum_{m=0}^{\infty} A^m f(a)(x - a)^m \tag{43}$$

is called *Taylor series* of f at a.

Many important theorems in complex analysis in Banach spaces can be shown to hold by reducing the problems to well known results of complex analysis in one or several complex variables.

Definition 4.3. *A map $f : U \to Y$ is called G-analytic if the mapping $\lambda \mapsto f(a + \lambda b)$ is analytic for all $a \in U$ and $b \in X$ on the open set $\{\lambda \in \mathbb{C} : a + \lambda b \in U\}$. It is called weakly analytic if $g \circ f$ is analytic for all $g \in Y'$, the dual space of Y.*

The following generalized Cauchy integral formula is useful.

Theorem 4.4. *Let $f : U \to Y$ be analytic, $a \in U$, $t \in X$ and $r > 0$ so that $a + \zeta t \in U$ for all $\zeta \in \overline{\mathcal{U}(0, r)} \subset \mathbb{C}$. Then for all $\lambda \in \mathcal{U}(0, r) \subset \mathbb{C}$ the Cauchy integral formula*

$$f(a + \lambda t) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \frac{f(a + \zeta t)}{\zeta - \lambda} \, \mathrm{d}\zeta \tag{44}$$

holds. Further,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(a + \lambda t) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=r} \frac{f(a + \zeta t)}{(\zeta - \lambda)^2} \, \mathrm{d}\zeta \tag{45}$$

is valid.

The integration paths are always positively oriented. An analogous formula exists for higher derivatives.

With the help of the generalized Cauchy integral formula the following relations between the different notions of differentiability can be shown:

Theorem 4.5.

$$\begin{aligned} f \text{ is analytic} \quad &\Leftrightarrow \quad f \text{ is continuous and } G\text{-analytic} \\ &\Leftrightarrow \quad f \text{ is locally bounded and } G\text{-analytic} \\ &\Leftrightarrow \quad f \text{ is weakly analytic.} \end{aligned}$$

After discussing the notion of analytic or holomorphic functions in the sense of power series, i.e. the point of view adopted by Weierstrass, we now turn to the notion of complex differentiability, i.e. the Riemannian point of view.

Definition 4.6. *A map $f : U \to Y$ is called differentiable (Fréchet-differentiable, complex differentiable or differentiable in norm) if for all $a \in U$ there exists a mapping $A \in \mathcal{L}(X, Y)$ so that*

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - A(h)\|}{\|h\|} = 0. \tag{46}$$

Alternatively, a map $f : U \to Y$ is called differentiable if for all $a \in U$ a mapping $A \in \mathcal{L}(X, Y)$ exists so that

$$f(a+h) - f(a) = A(h) + o(\|h\|) \tag{47}$$

for all h in a neighbourhood of zero. Here $r(h) = o(\|h\|)$ is an abbreviation for a mapping $r : \mathcal{U}(0) \subseteq X \to Y$ with $r(h)/\|h\| \to 0$ if $h \to 0$.

As a side remark we want to mention some further results.

(a) Every differentiable map $f : U \to Y$ is continuous.
(b) The mapping $g \mapsto g^{-1}$ is differentiable for every invertible map $g \in \mathcal{L}(Y)$. We will use this in connection with resolvents in section 5.
(c) The mapping $A \in \mathcal{L}(X, Y)$ of definition 4.6 is uniquely determined by f and a. It is called *derivative* of f in a and is often written in the form

$$Df(a) := A. \tag{48}$$

Every differentiable map $f : U \to Y$ induces a mapping $Df : U \to \mathcal{L}(X, Y)$. As in finite-dimensional Banach spaces the sum rule, product rule and chain rule are valid. In the proofs of section 5, where we generalize perturbation theory to coupling parameters in Banach spaces, we often use the equivalence between analyticity and complex differentiability.

Analogous to complex analysis in one complex variable it is shown that analyticity also implies that the map is infinitely often complex differentiable. Here higher derivatives are defined recursively, i.e. $f : U \to Y$ is *k-times differentiable* if f is a $(k-1)$-times differentiable mapping and if the $(k-1)$st derivative $D^{k-1}f : U \to \mathcal{L}(X^{k-1}, Y)$ is differentiable. A map is called *infinitely often differentiable* if it is k-times differentiable for each $k \in \mathbb{N}$ (we set $D^0 f = f$).

One can show that every m-linear mapping $D^m f(a) \in \mathcal{L}(X^m, Y)$ is symmetric for all $a \in U$. With this it is possible to prove the following important theorem.

Theorem 4.7. *For a map $f : U \to Y$ the following statements are equivalent:*

(a) f is analytic.
(b) f is complex differentiable.
(c) f is infinitely many times complex differentiable.

If one of these conditions is fulfilled,

$$D^m f(a) = m!\, A^m f(a) \tag{49}$$

holds.

With this we obtain the Taylor series of f in a

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m f(a)(x-a)^m \tag{50}$$

for all $x \in B(a, r) \subset U$ with a certain $r \geqslant 0$.

In section 5 we will use the notion of a differentiable map, which depends on a variable in a Cartesian product of Banach spaces. We want to show that the resolvent of the Hamilton operator $H(\beta)$,

$$\big(H(\beta) - \lambda\big)^{-1} \tag{51}$$

is jointly differentiable in (β, λ) with λ being an element of the resolvent set.

The Cartesian product $X \times Y$ of two Banach spaces X and Y becomes a Banach space by componentwise addition and scalar multiplication and with the norm $\|(x, y)\| := \|x\| + \|y\|$ for $(x, y) \in X \times Y$. Let $U \subset X \times Y$ be open and Z another complex Banach space. Then $f : U \to Z$ is called differentiable in analogy to definition 4.6 if for all $(a, b) \in U$ there exists a mapping $A \in \mathcal{L}(X \times Y, Z)$ so that

$$\lim_{(h,k)\to(0,0)} \frac{\|f(a+h, b+k) - f(a, b) - A(h, k)\|}{\|(h, k)\|} = 0 \tag{52}$$

or equivalently

$$f(a+h, b+k) = f(a, b) + A(h, k) + o(\|(h, k)\|) \tag{53}$$

holds. The mapping A is called derivative of f in $(a, b) \in U$ and is written as $Df(a, b) := A$.

To prove the differentiability of a mapping in, for example, two variables, one can introduce partial derivatives as in finite-dimensional spaces.

Theorem 4.8. *A map $f : U \subset X \times Y \to Z$ is differentiable if $f_1(x) := f(x, y)$ and $f_2(y) := f(x, y)$ are differentiable and if the derivatives $D_1 f(a, b) := Df_1(a)$ and $D_2 f(a, b) := Df_2(b)$ are continuous in (a, b). Then for each $(a, b) \in U$*

$$Df(a, b)(h, k) = D_1 f(a, b)h + D_2 f(a, b)k \tag{54}$$

holds for all $(h, k) \in X \times Y$.

5. Analytic perturbation theory in coupling parameters in Banach spaces

As the model Hamiltonian of section 1 suggests, we have to focus our attention on the perturbation theory of operators in infinite many complex coupling parameters. In the following we consider the β_i's as an element of a sequence space $l^p(\mathbb{C})$, $1 \leqslant p \leqslant \infty$. Of particular interest is the space l^∞, i.e. the sequences which are uniformly bounded.

As we have already remarked in section 4, we treat the sequence-space-valued coupling parameters in a more abstract way by regarding them as coupling parameters in a general complex Banach space.

In the first subsection we define analytic families and prove a generalization of a theorem of Kato and Rellich about the behaviour of isolated, non-degenerate eigenvalues and their eigenfunctions.

The second subsection deals with other notions of analytic families. We investigate in particular analytic families of type (A) and explore their relation to analytic families of the first subsection.

In the third subsection we show that relatively bounded perturbations are analytic families in our generalized sense. This then enables us to apply the machinery developed above to the model Hamiltonian of section 1.

5.1. Generalization of a theorem of Kato and Rellich

One of the goals of analytic perturbation theory is the representation of eigenvalues and eigenfunctions as power series in the complex coupling parameter. Therefore, the functions under discussion have to be analytic in the coupling parameter. One hopes that the eigenvalues and eigenfunctions are analytic if the corresponding Hamilton operator depends analytically on the coupling parameter in a certain way.

As we have explained in section 4 it is natural to investigate analytic mappings between a Banach space of coupling constants and a Banach space of operators, for example the bounded operators.

For unbounded (Hamilton) operators the situation is slightly different as the set of unbounded operators is not automatically a Banach space. It is, however, possible to metrize the set of closed operators and to define analytic families via a generalized convergence [Ka, p 197]. In this paper we use an equivalent definition according to [ReSi4, p 14]. In this approach analyticity of the corresponding resolvents is demanded, such that the problem is reduced to the case of bounded operators.

In the following 'β near β_0' always means that β is an element of a suitable neighbourhood of β_0. If not stated otherwise, the operators $T(\beta)$ are defined on a Banach space Y. X is always assumed to be a complex Banach space and $U \subset X$ to be open and connected.

Definition 5.1. *An operator-valued mapping $T(\cdot)$ on U is called an analytic family or an analytic family in the sense of Kato if and only if*

(a) For each $\beta \in U$, the operator $T(\beta)$ is closed and has a non-empty resolvent set, i.e. $\rho(T(\beta)) \neq \emptyset$.
(b) For every $\beta_0 \in U$ a $\lambda_0 \in \rho(T(\beta_0))$ exists so that $\lambda_0 \in \rho(T(\beta))$ if β is near β_0 and so that the resolvent $(T(\beta) - \lambda_0)^{-1}$ is an analytic operator-valued mapping of β in a neighbourhood of β_0.

The following investigations show that this definition is convenient and allows us to derive results about the behaviour of eigenvalues and eigenfunctions, such as, for example, the generalized theorem of Kato and Rellich.

For this we need the analyticity of the resolvent in both variables (β, λ).

Lemma 5.2. *If $T(\cdot)$ is an analytic family on U,*

$$\Gamma := \{(\beta, \lambda) : \beta \in U,\ \lambda \in \rho(T(\beta))\} \tag{55}$$

is open in $X \times \mathbb{C}$. The resolvent $(T(\beta) - \lambda)^{-1}$, which is defined on Γ, is analytic in (β, λ).

The proof, which is inspired by [ReSi4, p 14], can be found in [Sc]. It exploits the equivalence between analyticity and complex differentiability. The resolvent is analytic in $(\beta, \lambda) \in \Gamma$ if it is differentiable in each variable and if the partial derivatives are continuous in (β, λ) (theorem 4.8). In particular, we use the differentiability of the mapping $g \mapsto g^{-1}$.

With the help of this it is possible to generalize the theorem of Kato and Rellich of perturbation theory in one complex parameter [ReSi4, p 15] to coupling parameters in a complex Banach space.

Theorem 5.3. *Let $T(\cdot)$ be an analytic family in $\beta \in U$. Suppose that E_0 is an isolated, non-degenerate eigenvalue of $T(\beta_0)$, then the following is valid:*

(a) For β near β_0, there is exactly one isolated, non-degenerate point $E(\beta)$ in $\sigma(T(\beta))$ near E_0. $E(\beta)$ is an analytic map of β for β near β_0.
(b) There is an analytic eigenvector $\psi(\beta)$ of $T(\beta)$ for β near β_0.

Proof. If E_0 is a discrete eigenvalue of $T(\beta_0)$, one can find $r > 0$ so that $\{\lambda \in \mathbb{C} : |\lambda - E_0| \leqslant r\} \cap \sigma(T(\beta_0)) = \{E_0\}$. The circle $\{\lambda \in \mathbb{C} : |\lambda - E_0| = r\}$ is compact in $\mathbb{C}$ and a subset of $\rho(T(\beta_0))$. According to lemma 5.2 the set $\Gamma = \{(\beta, \lambda) : \beta \in U \subset X,\ \lambda \in \rho(T(\beta))\}$ is open in $X \times \mathbb{C}$. Therefore, $\delta > 0$ exists so that $\lambda \in \rho(T(\beta))$ if $|\lambda - E_0| = r$ and $\|\beta - \beta_0\| \leqslant \delta$. Then the resolvent $(T(\beta) - \lambda)^{-1}$ is analytic in (β, λ). Let $W := \{\beta \in X : \|\beta - \beta_0\| \leqslant \delta\}$. Then

$$P(\beta) = -(2\pi i)^{-1} \int_{|\lambda - E_0| = r} \big(T(\beta) - \lambda\big)^{-1} \,\mathrm{d}\lambda \tag{56}$$

exists for all $\beta \in W$ and is analytic in β if $\beta \in W \subset X$.

Since the eigenvalue E_0 of $T(\beta_0)$ is non-degenerate, the corresponding projector is one dimensional. Using a lemma in [ReSi4, p 14] we know that all projectors $P(\beta)$ are one dimensional if $\beta \in W$. According to theorem XII.6 in [ReSi4, p 13], which is also valid for operators in Banach spaces, there is exactly one non-degenerate eigenvalue $E(\beta)$ of $T(\beta)$ with $|E(\beta) - E_0| < r$ if $\beta \in W$.

Let ψ_0 be the corresponding eigenvector of E_0. Then $P(\beta)\psi_0 \neq 0$ if β is near β_0 because $P(\beta)\psi_0 \to \psi_0$ for $\beta \to \beta_0$. As $P(\beta)\psi_0$ is an eigenvector of the operator $T(\beta)$ in Y, we have for all $\varphi \in Y'$, the dual space of Y,

$$\begin{aligned} \varphi(P(\beta)\psi_0) &= \varphi\Big(\big(T(\beta) - E_0 - r\big)^{-1}\big(T(\beta) - E_0 - r\big)P(\beta)\psi_0\Big) \\ &= (E(\beta) - E_0 - r)\varphi\Big(\big(T(\beta) - E_0 - r\big)^{-1}P(\beta)\psi_0\Big). \end{aligned} \tag{57}$$

Hence

$$(E(\beta) - E_0 - r)^{-1} = \frac{\varphi\Big(\big(T(\beta) - E_0 - r\big)^{-1}P(\beta)\psi_0\Big)}{\varphi\big(P(\beta)\psi_0\big)} \tag{58}$$

and $(E(\beta) - E_0 - r)^{-1}$ is analytic if $\beta \in W$.

Define $\psi(\beta) := P(\beta)\psi_0$, then $\psi(\beta)$ is an analytic eigenvector of $T(\beta)$ if $\beta \in W$. □

5.2. Analytic families of type (A)

As the last theorem shows the notion of analytic families in the sense of Kato is also convenient for coupling parameters in general Banach spaces. As it is frequently difficult to verify directly that a given family of operators is an *analytic* family, other notions of analytic families are introduced.

In this paper we define analytic families of type (A) for operators depending on a parameter varying in a Banach space. It is then possible to show that analytic families of this type are analytic families in the more general sense of Kato. This is useful because it is usually easier to prove that a family of operators is analytic of type (A).

Definition 5.4. *For each $\beta \in U$, let $T(\beta) : \mathcal{D}(T(\beta)) \subset Y \to Y$ be a closed operator with non-empty resolvent set. $T(\cdot)$ is called an analytic family of type (A) if and only if*

(a) The domain $\mathcal{D} := \mathcal{D}(T(\beta))$ does not depend on $\beta \in U$.
(b) $T(\beta)\psi$ is an analytic map in $\beta \in U$ for all $\psi \in \mathcal{D}$.

In order to infer the more general property from this we prove the analyticity of the resolvent with the help of the 'strong' analyticity of the operators. In a first step we construct bounded operators from the closed operators and use the following lemma.

Lemma 5.5. *Let X, Y and Z be complex Banach spaces and let $U \subset X$ be open. If $\tilde{T}(\beta) \in \mathcal{L}(Z, Y)$ and if $\tilde{T}(\beta)\psi$ is analytic in $\beta \in U$ for all $\psi \in Z$, then $\tilde{T}(\beta)$ is analytic in $\beta \in U$.*

Proof. Let $\beta \in U$, $t \in X$ and $M = \{\zeta \in \mathbb{C} : \beta + \zeta t \in U\}$. Let $\Gamma \subset M$ be a mathematically positive oriented circle in ζ. The Cauchy integral formula (theorem 4.4) yields

$$\begin{aligned}
&\frac{1}{h}(\tilde{T}(\beta + (\zeta + h)t)\psi - \tilde{T}(\beta + \zeta t)\psi) - \frac{\mathrm{d}}{\mathrm{d}\zeta}\tilde{T}(\beta + \zeta t)\psi \\
&\qquad = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \left(\frac{1}{h}\left(\frac{1}{\zeta' - (\zeta + h)} - \frac{1}{\zeta' - \zeta}\right) - \frac{1}{(\zeta' - \zeta)^2}\right)\tilde{T}(\beta + \zeta' t)\psi \, \mathrm{d}\zeta' \\
&\qquad = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h}{(\zeta' - (\zeta + h))(\zeta' - \zeta)^2} \, \tilde{T}(\beta + \zeta' t)\psi \, \mathrm{d}\zeta'. \qquad (59)
\end{aligned}$$

As all analytic functions are G-analytic, $\tilde{T}(\beta)\psi$ is G-analytic (theorem 4.5). Hence $\tilde{T}(\beta+\zeta' t)\psi$ is continuous in $\zeta' \in M$. Since Γ is compact, for each $\psi \in Z$ a number C_ψ exists so that

$$\|\tilde{T}(\beta + \zeta' t)\psi\| \leqslant C_\psi \qquad (60)$$

for all $\zeta' \in \Gamma$. According to the uniform boundedness theorem a $C \in \mathbb{R}$ exists so that

$$\sup_{\zeta' \in \Gamma} \|\tilde{T}(\beta + \zeta' t)\| \leqslant C. \qquad (61)$$

Therefore, one obtains

$$\begin{aligned}
&\left\| \frac{1}{h}(\tilde{T}(\beta + (\zeta + h)t)\psi - \tilde{T}(\beta + \zeta t)\psi) - \frac{\mathrm{d}}{\mathrm{d}\zeta}\tilde{T}(\beta + \zeta t)\psi \right\| \\
&\qquad \leqslant \frac{1}{2\pi} C \|\psi\| \int_\Gamma \left| \frac{h}{(\zeta' - (\zeta + h))(\zeta' - \zeta)^2} \right| \mathrm{d}\zeta' \qquad (62)
\end{aligned}$$

hence the estimate

$$\begin{aligned}
&\left\| \frac{1}{h}(\tilde{T}(\beta + (\zeta + h)t) - \tilde{T}(\beta + \zeta t)) - \frac{\mathrm{d}}{\mathrm{d}\zeta}\tilde{T}(\beta + \zeta t) \right\| \\
&\qquad \leqslant \frac{1}{2\pi} C \int_\Gamma \left| \frac{h}{(\zeta' - (\zeta + h))(\zeta' - \zeta)^2} \right| \mathrm{d}\zeta' \qquad (63)
\end{aligned}$$

holds. The integral vanishes in the limit $h \to 0$. Therefore, $\tilde{T}(\beta)$ is G-analytic in $\beta \in U$. $\tilde{T}(\beta)$ is analytic if, in addition, $\tilde{T}(\beta)$ is locally bounded (theorem 4.5). The local boundedness follows from the uniform boundedness principle by means of the continuity of the mapping $\tilde{T}(\beta)\psi$. For each $\psi \in Z$ and every compact set $\Gamma \subset U$ a c_ψ exists so that $\|\tilde{T}(\beta)\psi\| \leqslant c_\psi$ is valid for all $\beta \in \Gamma$. Hence $c \in \mathbb{R}$ exists with $\sup_{\beta \in \Gamma} \|\tilde{T}(\beta)\| \leqslant c$. □

This yields the following important theorem (cf [Ka, p 375]).

Theorem 5.6. *Every analytic family of type (A) is an analytic family in the sense of Kato.*

Proof. Let $\beta_0 \in U \subset X$. Let $T(\beta) : \mathcal{D} \subset Y \to Y$ be an analytic family of type (A) in $\beta \in U$. Hence $T(\beta_0)$ is a closed operator with $\rho(T(\beta_0)) \neq \emptyset$. By introduction of the graph norm

$$|||\psi||| = \|\psi\| + \|T(\beta_0)\psi\| \qquad (64)$$

the domain $\mathcal{D}$ of this operator is converted into a Banach space

$$\tilde{D} := (\mathcal{D}, |\!|\!| \cdot |\!|\!|). \tag{65}$$

Let ι be the embedding operator from $\tilde{D}$ in Y. ι is bounded because $\|\iota\psi\| = \|\psi\| \leqslant |\!|\!|\psi|\!|\!|$ holds.

We now consider the operator $T(\beta)$ from $\tilde{D}$ to Y and call this operator $\tilde{T}(\beta)$,

$$\tilde{T}(\beta) : \tilde{D} \to Y, \tag{66}$$

$$\psi \mapsto \tilde{T}(\beta)\psi = T(\beta)\,\iota\,\psi. \tag{67}$$

$\tilde{T}(\beta)$ is a closed operator because $T(\beta)$ is closed and ι is continuous. $\tilde{T}(\beta)$ is defined on the whole $\tilde{D}$. Therefore, $\tilde{T}(\beta)$ is bounded according to the closed graph theorem, i.e. $\tilde{T}(\beta) \in \mathcal{L}(\tilde{D}, Y)$.

$\tilde{T}(\beta)\psi = T(\beta)\,\iota\,\psi = T(\beta)\psi$ is analytic in $\beta \in U$ for all $\psi \in \tilde{D}$. Therefore, $\tilde{T}(\beta)$ is analytic in $\beta \in U$ (lemma 5.5).

Let $\lambda_0 \in \rho(T(\beta_0))$. It has to be shown that λ_0 is an element of $\rho(T(\beta))$ and that $(T(\beta) - \lambda_0)^{-1}$ is an analytic map in β for β near β_0. The map $T(\beta_0) - \lambda_0 : \mathcal{D} \to Y$ is bijective because $\lambda_0 \in \rho(T(\beta_0))$. The same holds for $\iota : \tilde{D} \to \mathcal{D}$. Therefore, $(T(\beta_0) - \lambda_0)\iota = \tilde{T}(\beta_0) - \lambda_0\,\iota$ is invertible and

$$\left(\tilde{T}(\beta_0) - \lambda_0\,\iota\right)^{-1} \in \mathcal{L}(Y, \tilde{D}). \tag{68}$$

As the set of invertible, continuous and linear operators on Y is open (see, e.g., [Ta, p 9]),

$$\left(\tilde{T}(\beta) - \lambda_0\,\iota\right)^{-1} \in \mathcal{L}(Y, \tilde{D}) \tag{69}$$

and $(\tilde{T}(\beta) - \lambda_0\,\iota)^{-1}$ is analytic in β for β near β_0. Therefore,

$$(T(\beta) - \lambda_0)^{-1} = \iota(\tilde{T}(\beta) - \lambda_0\,\iota)^{-1} \tag{70}$$

is bounded and analytic in β (note that we modified the standard textbook proof which does not seem to be directly applicable to our more general situation). □

The inversion of this theorem is not valid as already has been shown by a counterexample for complex coupling parameters [Ka, p 376] or [ReSi4, p 20]. An analytic family in the sense of Kato can have a domain which depends on β.

5.3. Perturbation theory of Hamilton operators in infinitely many complex coupling parameters

In quantum mechanics the (Hamilton) operators are typically of the form

$$H(\beta) = H_0 + V(\beta). \tag{71}$$

Theorem 5.7. *Let X and Y be complex Banach spaces; let $U \subset X$ be open and connected. Suppose that H_0 is a closed operator on $\mathcal{D}(H_0) \subset Y$ and that for each $\beta \in U$, $V(\beta)$ is relatively H_0-bounded with H_0-bound smaller than one. Furthermore, let $V(\beta)\psi$ be analytic in $\beta \in U$ for all $\psi \in \mathcal{D}(H_0)$ and let the resolvent set $\rho(H(\beta))$ of*

$$H(\beta) = H_0 + V(\beta) \qquad \beta \in U \tag{72}$$

be non-empty. Then $H(\cdot)$ is an analytic family.

Proof. It is sufficient to prove that $H(\cdot)$ is an analytic family of type (A) (theorem 5.6). According to a well known stability theorem [Ka, p 190], $H(\beta)$ is a closed operator for all $\beta \in U$. The domain $\mathcal{D}(H(\beta)) = \mathcal{D}(H_0)$ does not depend on β. Because $V(\beta)\psi$ is an analytic map, $H(\beta)\psi$ is also analytic in $\beta \in U$ for all $\psi \in \mathcal{D}(H_0)$. Therefore, all conditions of an analytic family of type (A) are fulfilled. □

Remark 5.8. *Let H_0 be a self-adjoint operator on $\mathcal{D}(H_0) \subset \mathcal{H}$ and let V be relatively H_0-bounded with H_0-bound zero. Then the resolvent set $\rho(H_0 + V)$ is not empty.*

Proof. By definition we have

$$\|V\psi\| \leqslant a\|\psi\| + b\|H_0\psi\| \qquad \forall \psi \in \mathcal{D}(H_0). \tag{73}$$

As the H_0-bound of V is zero, b can be chosen arbitrarily small. The spectrum of the self-adjoint operator H_0 is real. Therefore, $\lambda \in \rho(H_0)$ with a sufficiently large imaginary part exists so that

$$a \sup_{E\in\sigma(H_0)} |E-\lambda|^{-1} + b \sup_{E\in\sigma(H_0)} |E||E-\lambda|^{-1} < 1 \tag{74}$$

holds. According to [Ka, pp 214, 272] λ is an element of $\rho(H_0 + V)$. □

In the first part of the paper we investigated Hamilton operators of the form

$$H(\beta) = H_0 + \sum_{i=1}^{\infty} \beta_i V_i. \tag{75}$$

$V(\beta)\psi = \sum_{i=1}^{\infty} \beta_i V_i \psi$ is analytic in $\beta = (\beta_1, \beta_2, \ldots) \in l^\infty(\mathbb{C})$ for all $\psi \in \mathcal{D}(H_0)$ because $\sum_{i=1}^{\infty} \beta_i V_i$ is continuous and linear in β for all $\psi \in \mathcal{D}(H_0)$. Bounded operators are relatively H_0-bounded with H_0-bound zero. Therefore, we obtain the following corollary.

Corollary 5.9. *Suppose H_0 to be a self-adjoint operator with $\mathcal{D}(H_0) \subset L^2(\mathbb{R}^m)$. Let V_i be symmetric, bounded operators in $L^2(\mathbb{R}^m)$ with $\|V_i\| \leqslant v$ for all $i \in \mathbb{N}$, which fulfil the finite intersection property. Let $\beta \in l^\infty(\mathbb{C})$. Then*

$$H(\beta) = H_0 + \sum_{i=1}^{\infty} \beta_i V_i \tag{76}$$

is an analytic family.

For Hamilton operators with an infinite sum of Stummel-class potentials we obtain a corresponding result.

Corollary 5.10. *Let $\rho < 4$ and $\beta \in l^p(\mathbb{C}), 1 \leqslant p \leqslant \infty$. Let $\{V_i\}_{i\in\mathbb{N}}$ be multiplication operators in $L^2(\mathbb{R}^m)$ so that the finite intersection property is fulfilled and so that $v_i \in M_\rho(\mathbb{R}^m)$ with $\sup_{i\in\mathbb{N}} M_{v_i,\rho} < \infty$. Then*

$$H(\beta) = -\Delta + \sum_{i=1}^{\infty} \beta_i V_i \tag{77}$$

is an analytic family. If in addition U is a symmetric, $-\Delta$-bounded operator with $-\Delta$-bound zero, the Hamilton operators $H(\beta) = -\Delta + U + \sum_{i=1}^{\infty} \beta_i V_i$ are an analytic family.

Proof. Since $-\Delta$ is a self-adjoint operator on $W_2(\mathbb{R}^m)$, $-\Delta + U$ is self-adjoint on $W_2(\mathbb{R}^m)$. In section 3 we proved that $V(\beta) := \sum_{i=1}^{\infty} \beta_i V_i$ is an element of the Stummel-class $M_\rho(\mathbb{R}^m)$. Therefore, $V(\beta)$ is relatively bounded with respect to $-\Delta + U$ with $(-\Delta + U)$-bound zero. □

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